Yvonne Choquet-Bruhat:

a Mathematician in Einstein’s Universe

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Online event to celebrate

Yvonne Choquet-Bruhat’s 100th Birthday,

International Society on General Relativity and Gravitation

January 24, 2024 via Zoom

Brief introduction to a few of YCB's many fundamental contributions to both the mathematical and the physical understanding of Einstein's theory of gravitation

3+1 decomposition;

Solving the constraints of the Cauchy problem;

« Stability of flat space »;

Positivity of mass in a neighborhood of Minkowski space;

New formulations of Einstein's equations which were important ingredients for being able to numerically simulate the motion and gravitational radiation of coalescing binary black holes.

A few words on YCB's early career

A « failed » physicist who constantly aimed at understanding the real universe through its theoretical physics description, by using, and perfecting, mathematical tools.

Father: Georges Bruhat, a physicist famous for his contributions to optics and for the many high-level physics text books he wrote

« In a sense my father had a significant influence on me, though it must be said that he had more interest in my brother [the mathematician François Bruhat]. He did not think I could become a renowned scientist. He thought I would become a good mother, and a gymnasium teacher. »

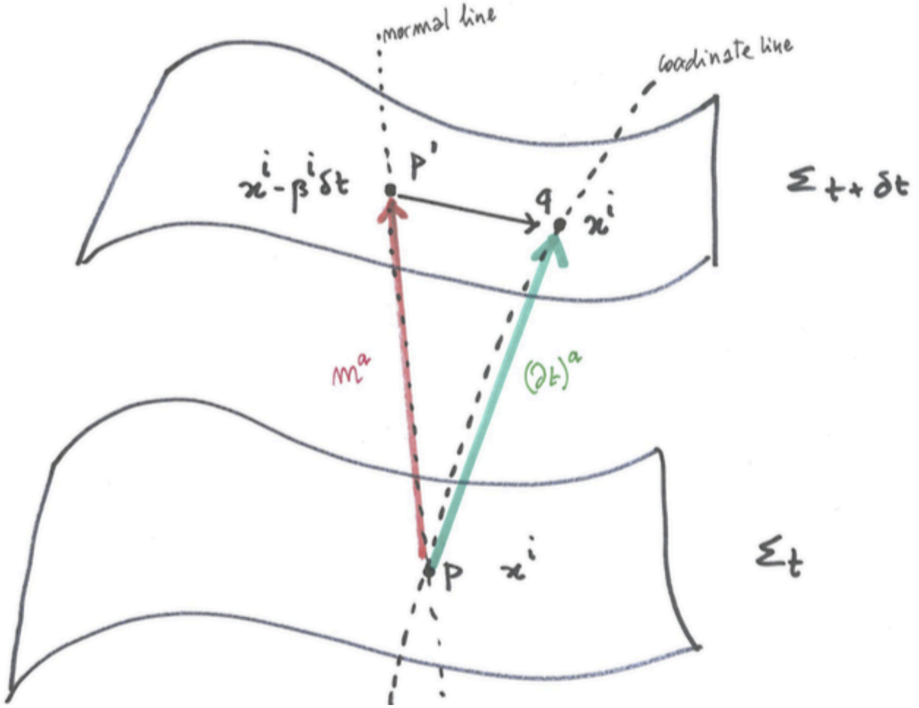
Mentors: Jean Leray and André Lichnerowicz

Upon a suggestion of **Jean Leray**, she chooses her main PhD thesis topic: to give the first mathematical proof of the existence (and well-posedness) of generic (non analytic) solutions of Einstein's vacuum field equations.

Reaction of her thesis advisor (**André Lichnerowicz**): « This is too difficult for a beginner. » However, Jean Leray encouraged, and helped, her.

3+1 decomposition of spacetime

slicing by a 1-parameter family of spacelike (Cauchy) surfaces



$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j ,$$

lapse N or alpha **shift N_i or beta^i** **3-metric h_ij, gammai_j or bar g_ij**

$$ds^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t)$$

CRAS 1948 and Journal of Rational Mechanics and Analysis 1956

Sur L'Intégration des Équations de la Relativité Générale

MADAME YVONNE FOURÈS-BRUHAT

Mémoire transmis par A. LICHNEROWICZ

$$\omega_4 = V\, dx^4,$$

$$\omega_i = a_{ij}\, dx^j + \lambda_i\, dx^4,$$

$$V^2 = \frac{1}{g^{44}} = \frac{g}{\bar{g}}.$$

5. Calcul des composantes du tenseur de Ricci: De l'expression du tenseur de Riemann on déduit après contraction: $R_{\alpha\beta} = -R_{\alpha\lambda\beta\lambda}$

$$R_{ih} = \bar{R}_{ih} - \frac{\overline{\nabla}_h(\partial_i V)}{V} - PP_{ih} - (\bar{Q}_{il}P_{lh} + Q_{hl}P_{li}) + \partial_4 P_{ih}$$

$$R_{4h} = \overline{\nabla}_h P - \overline{\nabla}_k P_{hk}$$

$$R_{44} = -\frac{\overline{\nabla}_h(\partial_h V)}{V} - P_{ih}P_{ih} + \partial_4 P.$$

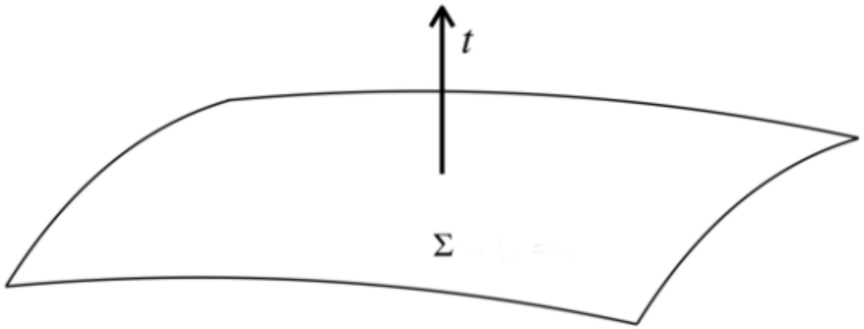
First, general 3+1 decomposition of Einstein's equations with lapse (V) and shift (lambda_i) before ADM's work

Darmois (1927) had only considered the case of unit lapse and zero shift

And first discussion of methods for constructing general solutions of the initial-value constraints. The beginning of a long, and important line of work

Lichnerowicz (1944) had considered less general cases

The problem of constraints on γ_{ij} and K_{ij}



$$R_\gamma - K_i^j K_j^i + (K_s^s)^2 = 2\rho$$
$$\nabla_j K_i^j - \nabla_i K_s^s = J_i$$

First and second fundamental forms

Darmois 1927

7. Indiquons la forme analytique des résultats. Posons

$$\Omega_{ik} = -\frac{1}{2}\frac{\partial g_{ik}}{\partial x_4}, \qquad g^{44} = g_{44} = +1.$$

Désignons par r_{ik} le tenseur analogue à R_{ik} de la variété S, par $[A_{ik}]e$ (crochet) la dérivation covariante dans S. On a

$$(8') \qquad R_{ik} = \frac{\partial \Omega_{ik}}{\partial x_4} + 2g^{mn}\Omega_{in}\Omega_{kn} - \Omega_{ik}(g^{uv}\Omega_{uv}) + r_{ik} = 0,$$

$$(9') \qquad R_{i4} = g^{hk}[\Omega_{ik}]_h - g^{uv}[\Omega_{uv}]_i = 0.$$

$$(10') \qquad R_{44} = g^{uv}\left(\frac{\partial \Omega_{uv}}{\partial x_4} + g^{ik}\Omega_{iv}\Omega_{ku}\right) = 0.$$

Des quatre conditions à vérifier, trois sont formées par (9'). La dernière résulte de la subtitution des valeurs (8'). Si l'on forme $R - 2R_{44}$, on a

$$(11) \qquad R - 2R_{44} = r - g^{ik,mn}\Omega_{ik,mn} = 0,$$

r est la courbure scalaire de S.

$$g^{ik,mn} = g^{im}g^{kn} - g^{in}g^{km},$$
$$\Omega_{ik,mn} = \Omega_{im}\Omega_{kn} - \Omega_{in}\Omega_{km}.$$

Early history on solving the constraints (cf YCB 2014)

$$\bar{g}_{ij} = \varphi^4 \gamma_{ij} \qquad \tilde{K}_{ij} = \varphi^2 (K_{ij} - \frac{1}{3}\bar{g}_{ij}\tau), \qquad \tau := \bar{g}^{ij} K_{ij}.$$

Racine 1934
Lichnerowicz 1944

$$D_i \tilde{K}^{ij} = \frac{2}{3}\varphi^6 \gamma^{ij} \partial_i \tau + \varphi^{10} J^j;$$

$$8\Delta_\gamma \varphi - R(\gamma)\varphi + |\tilde{K}|^2_\gamma \varphi^{-7} + (\rho - \frac{2}{3}\tau^2)\varphi^5 = 0$$

Racine 1934: if g_ij is conformally flat and maximal (tr K_ij=0)

$$\bar{g}_{ij} = \varphi^4 \delta_{ij}$$
$$\mathrm{tr} K = K^i_i = 0$$

$$\partial_j \tilde{K}^{ij} = 0$$
$$\Delta\varphi = -\frac{\tilde{K}^{ij}\tilde{K}_{ij}}{8\varphi^7}$$

YCB 1961: Elliptic system in harmonic coordinates for $\mathfrak{g}^{00}$ and $\mathfrak{g}^{0i}$

Vaillant-Simon 1969 constructs solutions near flat spacetime

York 1972, 1974 general solutions of the momentum constraint

YCB 1974 asympt flat solutions of Hamiltonian constr in Hölder spaces, later improved using weighted Sobolev spaces. [see her 2009 book OUP]

Beginnings of the Cauchy problem.

Yvonne Choquet-Bruhat

October 15, 2014

1410.3490 [grqc]

« Stability of flat space » (YCB+Deser 1973)

On the Stability of Flat Space

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Received December 14, 1972

It is shown that: (1) there exists, near flat space, a neighborhood of nonsingular asymptotically flat weak field solutions of the initial value equations of General Relativity; the solutions have physically appropriate generality; and (2) this neighborhood is complete and flat space is stable; every geometry representing a weak perturbation and satisfying the varied constraints is tangent to the space of solutions.

YCB+Deser 1973

We shall deal primarily with the simplest problem, in which the base geometry is the classical vacuum: flat space on R^3. It will be shown rigorously that near it, there *exists a complete neighborhood* of nonflat weak excitations, with physically interesting asymptotic behavior. These solutions have the generality to be expected, on physical grounds, of a massless tensor field. Vacuum is also *stable*: every perturbation satisfying the varied constraints is tangent to some curve of solutions lying in the neighborhood of flat space. This comparison provides a measure of the completeness of solution space with respect to that of allowed variations.

Positive mass results

1976 YCB+Marsden: First rigorous proof of positivity of mass for vacuum spacetimes near Minkowski, following Brill-Deser 1968, and using a critical point analysis in infinite dimensions

Solution of the Local Mass Problem in General Relativity

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Abstract. The local mass problem is solved. That is, in suitable function spaces, it is shown that for any vacuum space-time near flat space, its mass m is strictly positive. The relationship to other work in the field and some discussion of the global problem is given. Our proof is, in effect, a version of critical point analysis in infinite dimensions, but detailed L^p and Sobolev-type estimates are needed for the precise proof, as well as careful attention to the coordinate invariance group. For the latter, we prove a suitable slice theorem based on the use of harmonic coordinates.

2011 simplified spinorial proof (à la Witten) of energy positivity, and mass positivity (E > |P|), in any dimension

Problems with Using the 3+1 Formulation (YCB 1956, ADM 1962) for Solving Einstein Eqs.

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j,$$

First-order evolution system for h_ij and K_ij, given the lapse N and the shift N_i :

$$\frac{\partial h_{ij}}{\partial t} = -2NK_{ij} + N_{i|j} + N_{j|i},$$

$$\frac{\partial K_{ij}}{\partial t} = -N_{|ij} + N\left[{}^{(3)}R_{ij} + K_{ij}(\mathrm{tr}K) - 2K_{im}K_j{}^m\right]$$

$$+ \left[N^m K_{ij|m} + N^m_{|i} K_{jm} + N^m_{|j} K_{im}\right],$$

It was initially thought in Numerical Relativity that one could directly use such an evolution system. However, it led to many numerical instabilities when used as such.

Mathematically, it was on the contrary assumed to be « bad », i.e., non hyperbolic. However, YCB 2009 OUP showed that it was Leray-Ohya hyperbolic (« hyperbolique non strict ») , which ensures good causality properties (domain of dependence), albeit within non Sobolev-type functional spaces: Gevrey classes of non-analytic, but smooth functions.

1982-3 First (3+1)-type Hyperbolic Systems for Einstein's Eqs (YCB+ Ruggeri)

Hyperbolicity of the 3+1 System of Einstein Equations

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Abstract. By a suitable choice of the lapse, which in a natural way is connected to the space metric, we obtain a hyperbolic system from the 3+1 system of Einstein equations with zero shift; this is accomplished by combining the evolution equations with the constraints.

$$\frac{N}{\sqrt{h}} = \text{cst i.e.}\, g^{\mu\nu}\Gamma^0_{\mu\nu} = 0;\ \text{and}\, N_i = 0$$

harmonicity of the time coordinate

Later YCB+York 1996; generalized it to the non-zero shift case

YCB+ Ruggeri's 1983 3+1 hyperbolic system

$$\frac{N}{\sqrt{h}} = \text{cst i.e. } g^{\mu\nu}\Gamma^0_{\mu\nu} = 0; \text{ and } N_i = 0$$

$$a_i = \partial_i(\log[g^{1/2}])$$

$$\dot{g}_{ij} = -2k_{ij},$$

$$\Box k_{ij} + 3k_{h(i}R_{j)}{}^h - 2R_i{}^h{}_j{}^m k_{hm} + 2kR_{ij} - 2a_{(i}D_{j)}k - 4kD_i a_j$$
$$- k_{h(i}D_{j)}a^h + 2a^h D_{(i}k_{j)h} - a^h D_h k_{ij} - 2a^h D_{(i}k_{j)h} - 3ka_i a_j$$
$$- 4\alpha^{-2}k_{ih}k_{jm}k^{hm}$$

Later YCB+York 1996; generalized it to the non-zero shift case

descendants of these 3+1 hyperbolic systems have been used in Numerical Simulations of coalescing binary black holes

Descendants of YCB-Ruggeri 3+1 hyperbolic systems used in NR

time-harmonic slicing

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 K$$

Bona-Masso 1992
1+ log gamma slicing

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\frac{2}{\alpha} K$$

minimal-distorsion-like spatial gauges `a la Smarr-York 1978
Gamma-driver eqs of the shift
(van Meter+ 06, Gundlach+ 06)

conformal metric

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \mu_S \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk} - \eta \beta^i$$

BSSN formulation: **Shibata-Nakamura 1995; Baumgarte-Shapiro 1998**
Punctures: Brandt-Bruegmann 1997 represents black holes by punctures
Brill-Lindquist(1963)-like + Bowen-York (1980) initial data
moving punctures: Campanelli-Lousto-Maronetti-Zlochower 2006, Baker-Centrella-Choi-Koppitz-vanMeter 2006

Descendants of YCB 1952 harmonic-coords hyperbolic system used in NR

harmonic coordinates condition: $C_a^{(0)} \equiv -g_{ab}\Box x^b$ = 0

reduced Einstein eqs:

$$\frac{1}{2}g^{cd}g_{ab,cd} + g^{cd}{}_{(,a}g_{b)d,c} + \Gamma^c_{bd}\Gamma^d_{ac} = -8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right)$$

propagation of harmonic cond.

$$\Box C^a_{(0)} = -R^a{}_b C^b_{(0)}$$

generalized harmonic condition (Garfinkle 2002, Friedrich 2002)

$$C_a \equiv g_{ab}\left(H^a - \Box x^a\right) = 0.$$

constraint damping terms (Gundlach+ 2005, Pretorius 2005, Lindblom et al…)

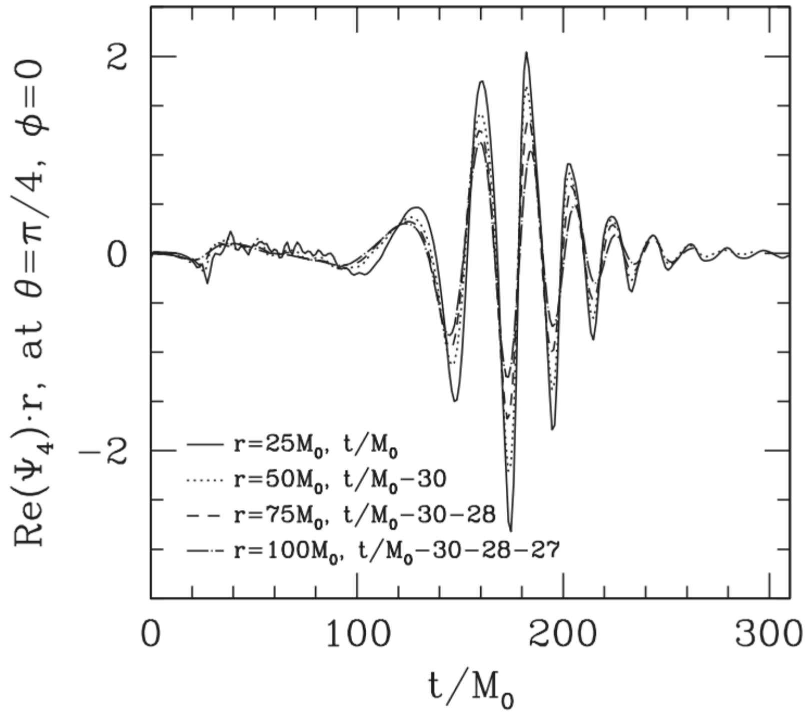
with kappa >0

$$\begin{aligned} \frac{1}{2}g^{cd}g_{ab,cd} \quad &+ \\ g^{cd}{}_{(,a}g_{b)d,c} + H_{(a,b)} \quad &- \quad H_d\Gamma^d_{ab} + \Gamma^c_{bd}\Gamma^d_{ac} \\ +\kappa\left[n_{(a}C_{b)}\right. \quad &- \quad \left.\frac{1}{2}g_{ab}n^d C_d\right] \\ &= \quad -8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right) \end{aligned}$$

$$\Box C^a = -R^a{}_b C^b + 2\kappa\nabla_b\left[n^{(b}C^{a)}\right]$$

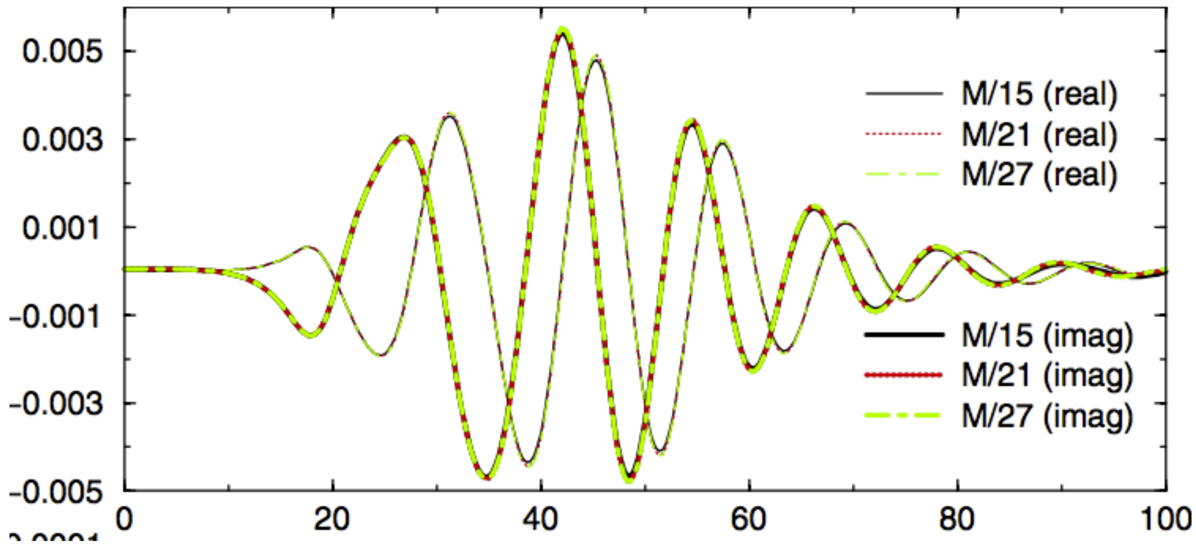
First succesfull numerical simulations of coalescing BBH

Pretorius 2005



using generalized harmonic with constraint damping

Campanelli + 2006



using BSSN with 1+log, gamma driver and moving punctures

Besides her many fundamental research contributions YCB has had a deep impact through the writing of many information-laden, influential books