From the Initial Value Problem for the Einstein Equations to Gravitational Waves

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Honoring

YVONNE CHOQUET-BRUHAT

THE INTERNATIONAL SOCIETY ON GENERAL RELATIVITY AND GRAVITATION IS

CELEBRATING HER 100TH BIRTHDAY AND CONTRIBUTIONS TO THE FIELD

January 24th, 2024

Yvonne Choquet-Bruhat

Photo of Yvonne Choquet-Bruhat in 2006. Copyright Notice: MFO, MFO License, see http://owpdb.mfo.de/.

Yvonne Choquet-Bruhat

Celebrating her 100th Birthday,

Contributions to General Relativity,

Mathematics and Physics.

- Yvonne Choquet-Bruhat: Short Biosketch
- Cauchy Problem for the Einstein Equations: 1952 article, its Implications and New Results
- Gravitational Waves
- Extensions of 1952 Article and Further Results
- Yvonne's Results on High-Frequency Waves and New Developments

Life

- Yvonne Choquet-Bruhat was born on December 29, 1923 in Lille (France) as the second of three children. Father Georges Bruhat, mother Berthe Hubert. Family moved to Paris, when Yvonne was two years old. Discussing science was part of family life.

Life

- Mother Berthe Hubert taught literature and philosophy. Father Georges Bruhat was a physicist, became vice-director of the École Normale Supérieure (ENS) in the mid 1930s. Both parents supported Yvonne's interest and education in mathematics and physics. Georges Bruhat was deported by the Nazis to a concentration camp in 1944, he died in 1945. This was a big shock for the family.
- ENS de Jeunes Filles (ENS for women), graduated in 1946.
- Work on research and thesis with André Lichnerowicz in 1951.
- Mentor and friend Jean Leray whom she met in 1947.
- 1947: Yvonne married Léonce Fourès.
- 1950: Daughter Michelle was born.
- 1961: Yvonne married Gustave Choquet.
- 1962: Son Daniel was born
- 1966: Daughter Geneviève was born.

Yvonne Choquet-Bruhat and Gustave Choquet 1974 in Berkeley

Life

- Professor, chair of celestial mechanics, at the Sorbonne in Paris.
- 1979: Yvonne became the first woman to be elected to the French Academy of Sciences.
- 2003: Dannie Heineman Prize for Mathematical Physics.
- 2015: Yvonne received the Grand-croix de l'ordre national de la Légion d'honneur (Grand Cross of the National Order of the Legion of Honor), the highest recognition in France.
- Elected member of the American Academy of Arts and Sciences.
- Feature of her character: Deal with difficulties through constructive solutions. In different moments throughout her professional as well as personal life.
- Paved the way for many women in mathematics and in the sciences.
- Inspiring researcher, teacher and person.
- Many deep and important results in General Relativity, Mathematics and Physics.
- Family, friends, human beings are important!

Einstein Equations and Spacetimes

Einstein Equations

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{g}_{\mu\nu}\, \mathbf{R} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}}\, \mathbf{T}_{\mu\nu}\ , \tag{1}$$

$\mathbf{R}_{\mu\nu}$ the **Ricci curvature tensor**,

$\mathbf{R}$ the **scalar curvature tensor**,

$\mathbf{g}_{\mu\nu}$ the **metric tensor** and

$\mathbf{T}_{\mu\nu}$ the **energy-momentum tensor**,

G the Newtonian constant of gravitation, and c the speed of light, (often set $G = c = 1$).

Einstein-Vacuum (EV) equations

$$\mathbf{R}_{\mu\nu} \quad = \quad \mathbf{0}\ . \tag{2}$$

Investigate dynamics of spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein's equations (1), resp. (2).

Initial Data and Constraint Equations

Classic Approach

Einstein's equations, for a Lorentzian metric $g_{\mu\nu},\ \mu,\nu = 0,\cdots,n$ can be rewritten as:

• a set of hyperbolic evolution equations,

and

• a set of constraint equations on an initial data surface S for a Riemannian metric $g_{ij},\ i,j = 1,\cdots,n$ and its initial time derivative k_{ij}.

G. Darmois (1929) noticed the existence of GR constraints in the analytic spacelike Cauchy problem for GR in normal coordinates (t time coordinate and x^i, $i = 1,2,3$ spatial coordinates)

$$g = -N(t,x^k)dt^2 + \underbrace{N_i}_{=0} dx^i dt + g_{ij}(t,x^k)dx^i dx^j$$

$N(t,x^k)$ denotes the lapse function and N_i the shift vector.

A. Lichnerowicz (1939) for the analytic spacelike Cauchy problem with zero shift N_i and arbitrary lapse N.

Cauchy Problem

Depending on the **matter and energy** present
$\Rightarrow$ **specify** corresponding **equations**.

Construct spacetime in an evolution problem of the Einstein equations.

An **initial data set** consists of

- a 3-dimensional manifold H,
- a complete Riemannian metric $\bar{g}_{ij}$,
- a symmetric 2-tensor k_{ij},
- and a well specified set of initial conditions corresponding to the matter-fields.
- The initial data are assumed to be smooth enough (finite differentiability, good Sobolev spaces). These have to satisfy the **constraint equations**.

A **Cauchy development** of an initial data set is
- a globally hyperbolic spacetime (M, g) verifying the Einstein equations
- and an imbedding $i : H \to M$ such that $i_*(\bar{g})$ and $i_*(k)$ are the first and second fundamental forms of $i(H)$ in M.

Constraint Equations - Spacelike

Initial data surface H, Riemannian metric $\bar{g}_{ij}$ with $i, j = 1, \cdots n$, symmetric tensor k_{ij} ("initial time derivative of the metric").

$$\begin{aligned} \bar{R} &= 16\pi T_{00} + |k|^2 - (trk)^2 \qquad (3) \\ \nabla_i k^i{}_j - \nabla_j k^i{}_i &= 8\pi T_{0j} \qquad (4) \end{aligned}$$

Important: Constraints propagate

shown by Darmois (in the analytic case) and by Choquet-Bruhat (in general).

Evolution Equations

The data evolves according to

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2\Phi k_{ij} + \mathcal{L}_X \bar{g}_{ij} \tag{5}$$

$$\frac{\partial k_{ij}}{\partial t} = (\bar{R}_{ij} + k_{ij}\mathrm{tr}k - 2k_{is}k^s_j)\Phi + \mathcal{L}_X k_{ij} - \bar{\nabla}_i \bar{\nabla}_j \Phi \tag{6}$$

with $\Phi := 1/\sqrt{-g^{ij}\partial_i t \partial_j t}$ denoting the lapse function and X the shift vector.

The time vector field is $T = \Phi N + X$ and $\mathcal{L}$ is the Lie derivative.

Some important open questions before 1952

Some important open questions before 1952 concerned

- Initial value problem for the Einstein equations
- Causality
- Gravitational waves and the finite speed of propagation of the gravitational field

$\Rightarrow$ Yvonne Choquet-Bruhat answered them all in 1952.

Discussions, ideas and partial results had been achieved before by several people, including Leray, Lichnerowicz, Darmois, Sobolev and others.

Well Posedness

Consider the Einstein vacuum equations

$$R_{\mu\nu} = 0 \ .$$

Theorem (Y. Choquet-Bruhat (Y. Fourès-Bruhat), 1952)

Let $(H, \bar{g}, k)$ be an initial data set satisfying the vacuum constraint equations in wave gauge. Then there exists a spacetime (M, g) locally in time satisfying the Einstein vacuum equations with $H \hookrightarrow M$ being a spacelike surface with induced metric $\bar{g}$ and second fundamental form k. The Lorentzian metric g in wave gauge depends continuously on the initial data. Its value at a given point p depends only on the past of p. This local solution is unique.

Important: The theorem assumes only *finite differentiability*, that is $C^5 \times C^4$.

Today: The initial data are in *Sobolev spaces*. No loss of derivatives.

Y. Fourès-Bruhat. *Théorème d'existence pour certain systèmes d'equations aux dérivées partielles nonlinéaires.* Acta Math. **88**. (1952). 141-225.

Issue: C^k class of initial data gives a loss of one degree of differentiability in the evolution.

Solution: Use Sobolev spaces. (Finite kth energy norms.) The initial data are in Sobolev spaces. No loss of derivatives.

Energy estimates: J. Schauder (1935) first employed energy estimates in different, simpler equation.

J. Leray 1953: Used energy estimates for quasilinear hyperbolic system.

S. Sobolev 1950s and 1960s: Establish and use energy estimates without loss of differentiability.

P.-A. Dionne 1962-1963: Use energy estimates without loss of differentiability, citing a paper by Sobolev in mid 1950s.

A.E. Fisher, J.E.Marsden 1970: Improve 1952 result, energy method.

T.J.R. Hughes, T.Kato, J.E. Marsden 1977: Improve 1952 result, energy method.

Proof

Ideas of the Proof: Use wave coordinates (often called harmonic coordinates even though the metric is Lorentzian).

By definition, wave coordinates x^α satisfy the wave equation

$$\Box_g x^\alpha = 0 \ .$$

This is equivalent to the connection coefficients of these local wave coordinates satisfying

$$g^{mn}\Gamma^\alpha_{mn} = 0 \ .$$

The EV equations in wave coordinates become

$$\Box_g g_{\alpha\beta} = N_{\alpha\beta}(g, \nabla g) \tag{7}$$

with $N_{\alpha\beta}(g, \nabla g)$ denoting nonlinear terms with quadratics in ∇g.

Thus, we have a system of quasilinear wave equations. Study Cauchy problem for the reduced Einstein equations (7). Combine this with applying the domain of dependence theorem. $\Rightarrow$ Prove the theorem above. Used Kirchhoff-Sobolev parametrix.

Recall: $\Box_g \Phi = g^{\mu\nu}\nabla_\mu(\partial_\nu \Phi)$.

Gravitational Waves

Y. Fourès-Bruhat. *Théorème d'existence pour certain systèmes d'equations aux dérivées partielles nonlinéaires.* Acta Math. **88**. (1952). 141-225.

The Yvonne Choquet-Bruhat Local Existence and Uniqueness Theorem of 1952 proves that

- Gravitational waves exist for the nonlinear Einstein equations,
- and they propagate at the speed of light.

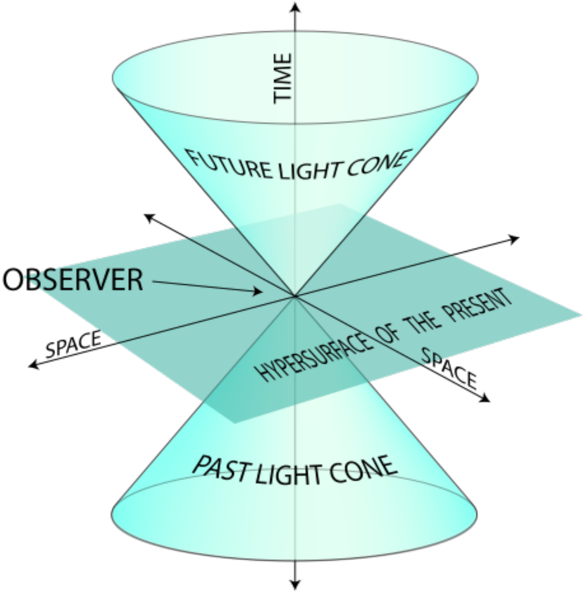
• Before that, there was confusion about the reality of gravitational waves.

• Albert Einstein had found wave solutions for the linearized Einstein equations 1916, 1918.

• Before 1952 there was the question: Do gravitational waves exist in the nonlinear theory?

$\Rightarrow$ Yes, they do! Y. Choquet-Bruhat's proof of 1952.

Causality and Light Cone



Further Studies on Radiation

- 1960s : Studies of gravitational waves (various authors). Use null hypersurfaces to describe gravitational radiation.
- On what is called the "Bondi energy" and "peeling properties" of the curvature at null infinity:
  Pioneering studies by Pirani (1957), Trautman (1958), Robinson and Trautman, Bondi and van der Burg and Metzner, Sachs, Penrose, Newman and Penrose in the 1960s.
- 1960s and 1970s : Choquet-Bruhat, Müller zum Hagen and Seifert : Earlier work on characteristic initial value problem.
- 1990 : Alan Rendall solves the local characteristic initial value problem for the Einstein vacuum equations.
- Since 1990 : Null hypersurfaces have played a crucial role in research related to gravitational radiation.
- Today : Peeling in the Christodoulou-Klainerman [CK] picture and in the Newman-Penrose [NP] picture.
- Today : PN and related topics. Many contributors such as Landau, Lifshitz, Thorne, Will, Blanchet, Damour, Epstein, Wagoner, Will, Wiseman, and many more.

Gravitational Waves

• Indirect detection of gravitational waves from observations of the Hulse-Taylor pulsar: Thibault Damour and Nathalie Deruelle computed the decrease in the orbital period of the binary system.

• Direct detection of gravitational waves: Luc Blanchet and Thibault Damour described the motion of two black holes approaching each other, and Thibault Damour with Alessandra Buonanno the final merger.

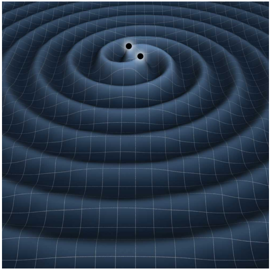
Gravitational Radiation

Fluctuation of curvature of the spacetime

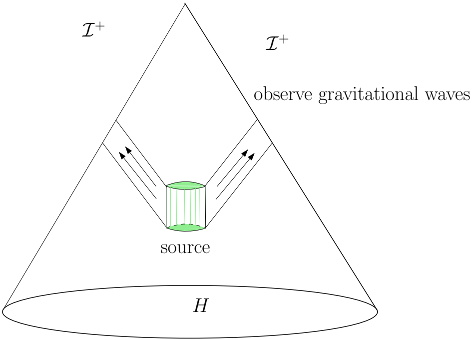
propagating as a wave.

Gravitational waves:

Localized disturbances in the geometry propagate at the speed of light, along outgoing null hypersurfaces.



Picture: Courtesy of NASA.



GW: Measurements - Beginning of a New Era

- LIGO detected gravitational waves from binary black hole mergers for the first time in September 2015.
- Several times since then.
- LIGO and VIRGO together observed gravitational waves from a binary neutron star merger in 2017. At the same time, several telescopes registered data.

1952 Result

The Yvonne Choquet-Bruhat Local Existence and Uniqueness Theorem of 1952 was a breakthrough in mathematics and in physics.

- Write the Einstein equations as quasilinear wave equations for the metric.
- Establish and solve the Cauchy problem for the Einstein equations in physical settings.
- Finite differentiability.
- Causality.
- Prove that gravitational waves exist for the nonlinear Einstein equations and
- they propagate at the speed of light.

Extended and using energy estimates and Sobolev spaces, this pioneering work has been and will be

- at the beginning of studies of global problems,
- and modern research investigating the dynamics of the Einstein equations.

1952 Result

The main results are established for a larger class of equations beyond Einstein equations, namely for a class of second order hyperbolic partial differential equations (pde). The application to general relativity concerns a "simpler" subclass of these pde.

4 spacetime dimensions

$$A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0 \tag{8}$$

with W_s, $s = 1, \cdots n$ the unknowns, and x^λ, $\lambda = 1, \cdots, 4$ the spacetime coordinates. The functions f_s and the coefficients $A^{\lambda\mu}$ depend on the unknowns and their first derivatives. Application to GR: $A^{\lambda\mu}$ depends only on the unknowns but not their first derivatives.

The Choquet-Bruhat local existence and uniqueness theorem has been generalized to hold for many coupled Einstein-matter systems. Many of these works were done by Choquet-Bruhat herself.

- Fourès-Bruhat 1951: Einstein-Maxwell in $D = 4$
- Fourès-Bruhat: Einstein equations in $D > 4$
- Fourès-Bruhat 1958: Einstein + perfect fluids. Leray-hyperbolic
- Bruhat 1958: Einstein + charged fluids with zero conductivity
- Bruhat 1961: Einstein + charged fluids with infinite conductivity
- Choquet-Bruhat 1966: Einstein + charged fluids with finite conductivity (not strictly hyperbolic (Leray-Ohya)), Gevrey classes
- Choquet-Bruhat: Einstein-Vlasov
- Choquet-Bruhat: Einstein + Yang-Mills

Theorem (Y. Choquet-Bruhat, R. Geroch 1969)

Let $(H, \bar{g}, k)$ be an initial data set satisfying the vacuum constraint equations. Then there exists a unique, globally hyperbolic, maximal spacetime (M, g) satisfying the Einstein vacuum equations with $H \hookrightarrow M$ being a Cauchy surface with induced metric $\bar{g}$ and second fundamental form k.

Global Solutions - Stability of Minkowski Space

The celebrated result by Sergiu Klainerman and Demetrios Christodoulou, 1991, proving the global nonlinear stability of Minkowski spacetime.

Theorem [D. Christodoulou and S. Klainerman for EV (1991)] (simplified version)

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

Generalizations of this Result:

[N. Zipser for EM (2000)] Generalization for Einstein-Maxwell case.

[L. Bieri for EV (2007)] Generalization in the Einstein-vacuum case.

All the above: geometric-analytic proofs.

Long list of other results and partial results. Works by many authors: Including but not complete: Y. Choquet-Bruhat, H. Friedrich, R. Geroch, P. Hintz, S. Hawking, H. Lindblad, F. Nicolò, R. Penrose, I. Rodnianski, A. Vasy, and more.

Asymptotic Flatness

(Christodoulou-Klainerman) Strongly asymptotically flat initial data set: an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and k_{ij} are:

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\, \delta_{ij} + o_4\,(r^{-\frac{3}{2}}) \tag{9}$$

$$k_{ij} = o_3\,(r^{-\frac{5}{2}})\,, \tag{10}$$

where M denotes the mass.

(B) Asymptotically flat initial data set: an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\frac{1}{2}}) \tag{11}$$

$$k_{ij} = o_2\,(r^{-\frac{3}{2}})\,. \tag{12}$$

Continue New Results

- Black Hole Formation 2008: D. Christodoulou shows that closed trapped surfaces, and eventually black holes, form in the Cauchy development of initial data, which are arbitrarily dispersed, if the incoming energy per unit solid angle in each direction in a suitably small time interval is large enough.
- Breakdown Criteria in GR 2010: S. Klainerman, I. Rodinanski. See Sergiu's talk.
- Proof of L^2 Curvature Conjecture 2012: S. Klainerman, I. Rodinanski, J. Szeftel. See Sergiu's talk.
- Stability Proofs: Minkowski, Kerr (recent results by Klainerman-Szeftel, (Schwarzschild) Dafermos-Rodnianski-Holzegel-Taylor. See Sergiu's talk.
- Results on Gravitational Radiation: Many authors (including the speaker) using spacetimes constructed via solving the Cauchy problem for the Einstein equations (for large data) and deriving information about radiation at future null infinity.
- Numerical Relativity: Base for numerical relativity. Generalized harmonic coordinates. See Thibault's talk.
- Many more $\cdots$

High-Frequency Waves

- High-Frequency Waves: Since 1960s, Yvonne Choquet-Bruhat produced various results on waves propagating in and interacting with a background, where the wavelengths of the waves are much shorter in comparison with the length scale of variation of the background. This includes gravitational waves propagating in a cosmological background.

  Weak progressive gravitational wave on spacetime $(M, \bar{g})$ given by C^2 Lorentzian metric g as follows

  $$g_{ab}(x, \omega\phi(x)) \quad = \quad \bar{g}_{ab}(x) + \omega^{-2}\hat{g}_{ab}(x, \omega\phi(x)) \tag{13}$$

  g is an asymptotic solution of the Einstein vacuum equations if Ricci of g satisfies

  $$R_{\alpha\beta} = \omega^{-2}\mathcal{R}_{\alpha\beta}(x, \omega)$$

  with $\mathcal{R}(x, \omega)$, ($x \in M$, $\omega \in \mathbb{R}$), uniformly bounded in $\omega > 0$ for some norm on M. Correspondingly define for Einstein-matter systems.

High-Frequency Waves

Parameter ω is called the frequency of the perturbation.

The surfaces $\phi = \mathrm{const}$ are wavefronts, since in the large ω limit, the waves vary rapidly in the direction perpendicular to them.

Strong High-Frequency Waves

- Strong High-Frequency Waves: Since 1960s, Yvonne Choquet-Bruhat produced various results on strong high-frequency waves.

Strong high-frequency gravitational wave in vacuum is a Lorentzian metric g as follows

$$g_{ab}(x, \omega\phi(x)) \quad = \quad \bar{g}_{ab}(x) + \omega^{-1}\tilde{g}_{ab}(x, \omega\phi(x)) + \omega^{-2}\hat{g}_{ab}(x, \omega\phi(x))$$

where the part $\bar{g}_{ab}$ is non-oscillatory, and the remainder is varying rapidly.

g is an asymptotic solution of order p of the Einstein vacuum equations if $\omega^p Ricci(g(x, \omega\phi(x)))$ remains uniformly bounded as $\omega \to \infty$.

High-Frequency Waves

Yvonne Choquet-Bruhat extended existing approaches and developed new mathematical method to deal with these physical problems. For more details, see her book *"General Relativity and the Einstein Equations"*.

New Results:

• L. Bieri, D. Garfinkle, N. Yunes 2017: Examine gravitational radiation in ΛCDM cosmology. Universe highly inhomogeneous, gravitational radiation treated in the short wavelength approximation.

• C. Huneau, J. Luk 2017: High-frequency backreaction for the Einstein equations under polarized $\mathbb{U}(1)$ symmetry.

• J. Luk, I. Rodnianski 2020: High-frequency limits without symmetry assumptions.

• A. Touati 2023: Extend Yvonne Choquet-Bruhat's result of 1969: Construct high-frequency solutions to the Einstein vacuum equations without any symmetry assumptions. In the limit this family approaches a fixed background solution of the Einstein-null dust system (backreaction phenomenon).

Some further works by Yvonne include

- *Y.Choquet-Bruhat, P.T.Chruściel, and J.M.Martín-García 2009-11:* Works on the light cone. Including: Area of cross-sections of light cones, in spacetimes satisfying suitable energy conditions, is smaller than or equal to that of the corresponding cross-sections in Minkowski, or de Sitter, or anti-de Sitter spacetime. The equality holds if and only if the metric coincides with the corresponding model in the domain of dependence of the light cone. In 2011: Prove existence and uniqueness theorem for the Cauchy problem on a characteristic cone for the Einstein vacuum equations.

Many books, they include

- *Y.Choquet-Bruhat:* 800-page-volume "General Relativity and the Einstein Equations"

- *Y.Choquet-Bruhat, C. DeWitt:* "Analysis, Manifolds and Physics"

Cécile DeWitt-Morette and Yvonne Choquet-Bruhat at IHES in 2015.

Credit: Chris DeWitt

Joyeux Anniversaire Yvonne!

Happy Birthday Yvonne!